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Adomian decomposition method for solving a Generalized Korteweg – De Vries equation with boundary conditions

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Abstract. In this paper, the Adomian decomposition method for the approximate solution of generalized Korteweg de Vries equation with boundary conditions is implemented. By using this method, the solution is calculated in the form of power series. The method does not need linearization, weak nonlinearity or perturbation theory. By using Mathematica Program, Adomian polynomials of the obtained series solution have been evaluated.

Keywords: Adomian decomposition method; Korteweg–deVries equation.

1. Introduction

The concept of Solitary Wave was introduced to the budding science of Hydrodynamics in 1834 by Scott Russell. In the following decades after him the Solitary Wave of Translation was briefly mentioned by various mathematicians. During the latter half of the nineteenth century, a mild controversy arose centered on the statement. This was quoted from the basic paper written by Korteweg and de Vries in 1895. Based on the result of their work was the derivation of an equation, which later became known as the Korteweg – de Vries (KdV) equation. It describes the evolution of long water waves down a canal of rectangular cross section[1,2]. The non linear KdV equation has been the focus of considerable studies by[3-8]. A feature common to all these methods is that they are using the transformations to reduce the equation into more simple equation than solve it. The main difficulty in using such method is that the obtained integral equation cannot always be solved in terms of the simple function. Unlike classical techniques, the nonlinear problems are solved easily and elegantly, without linearizing the problem by using the Adomian's decomposition method (ADM for short)[9-11].

In this paper we are going to use the ADM to solve The Generalized Korteweg – De Vries equation (GKdV):

$$u_t + \epsilon\, u^p u_x + \mu u_{xxx} = 0 \tag{1}$$

Where $p = 1, 2, ...$

Then we are going to use this method to solve the Modified Korteweg–De Vries equation (MKdV):

$$u_t + \epsilon\, u^2 u_x + \mu u_{xxx} = 0 \tag{2}$$

and Korteweg–De Vries Equation (KdV):

$$u_t + \epsilon\, u u_x + \mu u_{xxx} = 0 \tag{3}$$

the generalized homogeneous KdV equations with boundary conditions will be handled more easily, quickly, and elegantly by the ADM rather than the traditional methods.

2. Analysis of the Method

In the preceding section we have discussed particular devices of the general type of the GKdV equation. For purposes of the ADM, in this study we shall consider the equation:

$$u_t + \epsilon u^p u_x + \mu u_{xxx} = 0 \tag{4}$$

Where $p = 1,2,3, ...$

The solution of which is to obtain subject to the boundary condition:

$$u(a,t) = g_1(t), u_x(a,t) = g_2(t), u_{xx}(b,t) = g_3(t) \tag{5}$$

Where $g_i(t)$ corresponds to the data from the exact solution.

Let us consider the GKdV equation (4) in an operator form:

$$L_t u + \epsilon N u + \mu L_x u = 0 \tag{6}$$

Where the notation $Nu = u^p u_x$ symbolizes the nonlinear term, the notation $L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial^3}{\partial x^3}$ symbolizes the linear differential operator.

Assuming the inverse of the operator L_x^{-1} exists and it can conveniently be taken as the definite integral with respect to x; that is $L_x^{-1} = \int_a^x \int_a^x \int_b^x (\cdot)\, dx\, dx\, dx$.

Thus, applying the inverse operator L_x^{-1} to (2) yields:

$$\mu L_x^{-1} L_x u = -\epsilon L_x^{-1} N u - L_x^{-1} L_t u \tag{7}$$

$$L_x^{-1} L_x u = \int_a^x \int_a^x \int_b^x u_{xxx}\, dx\, dx\, dx = u(x,t) - \frac{1}{2} x^2 u_{xx}(b,t) - x u_x(a,t) + a x u_{xx}(b,t) - u(a,t) + \frac{1}{2} a^2 u_{xx}(b,t) + a u_x(a,t) + a^2 u_{xx}(b,t) \tag{8}$$

thus

$$u(x,t) = \frac{1}{2} x^2 g_2(t) + x g_1(t) - a x g_2(t) + u(a,t) - \frac{1}{2} a^2 g_2(t) - a g_1(t) - a^2 g_2(t) - \frac{\epsilon}{\mu} L_x^{-1} N u - \frac{1}{\mu} L_x^{-1} L_t u \tag{9}$$

we assign the zero component by:

$$u_0(x,t) = \frac{1}{2} x^2 g_2(t) + x g_1(t) - a x g_2(t) + u(a,t) - \frac{1}{2} a^2 g_2(t) - a g_1(t) - a^2 g_2(t) \tag{10}$$

A sum of components defined by the decomposition series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{11}$$

and the nonlinear term $Nu = u^p u_x$, we expressed in the form of A_n Adomain's polynomials; thus

$$Nu = u^p u_x = \sum_{n=0}^{\infty} A_n \tag{12}$$

In this specific nonlinearity, we use the general form of formula for A_n polynomials [3]:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\phi\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right]_{\lambda=0}, n \geq 0 \tag{13}$$

This formula is easy to set computer code to get as many polynomial as well as explicit solutions, for the reader to follow easily, we can give the first few Adomian polynomials these are:

$$A_0 = u_0^p (u_0)_x$$
$$A_1 = p u_0^{p-1} u_1 (u_0)_x + u_0^p (u_1)_x \tag{14}$$
$$A_2 = p u_0^{p-1} u_2 (u_0)_x + \frac{p(p-1)}{2} u_0^{p-2} u_1^2 (u_1)_x + p u_0^{p-1} u_1 (u_1)_x + u_0^p (u_2)_x$$
$$\begin{aligned} A_3 = {} & p u_0^{p-1} u_3 (u_0)_x + p(p-1) u_0^{p-2} u_1 u_2 (u_0)_x \\ & + \frac{p(p-1)}{2} u_0^{p-2} u_1^3 (u_0)_x + p u_0^{p-1} u_2 (u_1)_x \\ & + p u_0^{p-1} u_1 (u_2)_x + u_0^p (u_3)_x \end{aligned}$$

and so on, the other polynomials can be constructed in a similar way.

The remaining components $u_n(x,t), n \geq 1$, can be completely determined so that each term is computed by using the previous term, since u_0 is known,

$$u_1 = \frac{1}{\mu}[-\epsilon L_x^{-1}(A_0) - L_x^{-1} L_t (u_0)]$$
$$u_2 = \frac{1}{\mu}[-\epsilon L_x^{-1}(A_1) - L_x^{-1} L_t (u_1)] \tag{15}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_n = \frac{1}{\mu}[-\epsilon L_x^{-1}(A_{n-1}) - L_x^{-1} L_t (u_{n-1})]$$

As a result, the series solution is given by:

$$u(x,t) = u_0 - \frac{1}{\mu} \sum_{n=1}^{\infty} [\epsilon L_x^{-1}(A_{n-1}) + L_x^{-1} L_t (u_{n-1})] \tag{16}$$

3. Convergence Analysis of the Method

The convergence of the decomposition series has been investigated by several authors[12 - 13]. In this work we will discuss the convergence analysis[14] of ADM.

Let us consider the Hilbert space, defined by: $H = L^2\big((\alpha,\beta) \times [0,T]\big)$ the set of applications:

$$u: (\alpha,\beta) \times [0,T] \to \mathbb{R} \text{ with } \int_{(\alpha,\beta)\times[0,T]} u^2(x,s)\, ds\, d\tau < +\infty$$

And the following scalar product:

$$u(u,v)_H = \int_{(\alpha,\beta)\times[0,T]} u^2(x,s)\, ds\, d\tau$$

and:

$$\|u\|_H^2 = \int_{(\alpha,\beta)\times[0,T]} u^2(x,s)\, ds\, d\tau$$

the associated norm.

$$L(u) = u_t, R(u) = \epsilon u^p u_x, N(u) = u_{xxx}$$

Let us write our problem in an operator form:

$$L(u) + R(u) + N(u) = 0$$

and we set:

$$T(u) = -R(u) - N(u)$$

Let us consider the hypotheses:

- (H_1) $(T(u) - T(v), u - v) \geq k\, \|u - v\|^2$; $k > 0, \forall\, u, v \in H$
- (H_2) whatever may be $M > 0$, there exists a constant $C(M) > 0$ such that for $u, v \in H$ with $\|u\| \leq M, \|v\| \leq M$ we have:

$(T(u) - T(v), w) \leq C(M)\, \|u - v\|\, \|w\|$ for every $w \in H$

with the above hypotheses (H_1) and (H_2), Cherruault[12]; Mavoungou and Cherruault[13] have proved convergence of the method by using ideas developed by Cherruault[12] when applied to general partial differential equation.

Now we can verify the convergence hypotheses (H_1); *i.e.* there exists constant $k > 0$ such that for $u, v \in H$, we have: $(T(u) - T(v), u - v) \geq k\, \|u - v\|^2$.

$$\begin{aligned} T(u) - T(v) &= -\epsilon \left(u^p \frac{\partial}{\partial x} u - v^p \frac{\partial}{\partial x} v \right) - \left(\frac{\partial^3}{\partial x^3} u - \frac{\partial^3}{\partial x^3} v \right) \\ &= -\epsilon \left(\frac{1}{p+1} \right) \frac{\partial}{\partial x} (u^{p+1} - v^{p+1}) - \frac{\partial^3}{\partial x^3} (u - v) \end{aligned}$$

$$\begin{aligned}(T(u) - T(v), u - v) \\ &= \left(-\epsilon \left(\frac{1}{p+1}\right)\frac{\partial}{\partial x}(u^{p+1} - v^{p+1}), u - v \right) \\ &+ (- \frac{\partial^3}{\partial x^3}(u - v), u - v)\end{aligned}$$

Notice that $\frac{\partial}{\partial x}, \frac{\partial^3}{\partial x^3}$ are differential operators in, then there exist real constant $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}\left(\frac{\partial}{\partial x} (u^{p+1} - v^{p+1}), u - v \right) &\leq \delta_1 \|u^{p+1} - v^{p+1}\| \times \|u - v\| \\ &\leq \delta_1 \|(u - v) (u^p + u^{p-1}v + u^{p-2}v^2 + \ldots + v^p)\| \\ &\times \|u - v\| \\ &\leq \delta_1 \|u - v\| \\ &\times \| (u^p + u^{p-1}v + u^{p-2}v^2 + \ldots + v^p)\| \times \|u - v\| \\ &\leq (p + 1) \delta_1 M^p \|u - v\|^2\end{aligned}$$

$$- \left(\frac{\partial}{\partial x} (u^{p+1} - v^{p+1}), u - v \right) \geq (p + 1) \delta_1 M^p \|u - v\|^2$$

Where $\|u\| \leq M, \|v\| \leq M.$

Similarly

$$\left(\frac{\partial^3}{\partial x^3} (u - v), u - v \right) \leq \delta_2 \|u - v\|^2$$

$$- \left(\frac{\partial^3}{\partial x^3} (u - v), u - v \right) \geq \delta_2 \|u - v\|^2$$

$$\begin{aligned}(T(u) - T(v), u - v) \\ &\geq \epsilon \left(\frac{1}{p+1}\right) (p + 1) \delta_1 M^p \|u - v\|^2 + \delta_2 \|u - v\|^2\end{aligned}$$

Sitting $= p\, \delta_1\ M^p + \delta_2$, we obtain hypothesis (H_1).

- We can prove the hypothesis (H_2), i.e. $\forall\, M > 0$, $\exists\, C(M) > 0$ such that for $u, v \in H$ with $\|u\| \leq M, \|v\| \leq M$ we have:

$$(T(u) - T(v), w) \leq C(M) \|u - v\| \|w\|, \forall\, w \in H.$$

Indeed we have:

$$
\begin{aligned}
&(T(u) - T(v), w) \\
&\qquad = \left(-\epsilon \left(\frac{1}{p+1}\right)\frac{\partial}{\partial x}(u^{p+1} - v^{p+1}) - \frac{\partial^3}{\partial x^3}(u-v), w\right)
\end{aligned}
$$

$$\le \left(-\epsilon\left(\frac{1}{p+1}\right)\frac{\partial}{\partial x}(u^{p+1} - v^{p+1}), w\right) + \left(-\frac{\partial^3}{\partial x^3}(u-v), w\right)$$

$$\le \left|-\epsilon\left(\frac{1}{p+1}\right)\right|(p+1)\,\delta_1\, M^p\,\|u-v\|\,\|w\| + \delta_2\|u-v\|\,\|w\|$$

$$= (\epsilon\,\delta_1\, M^p + \delta_2)\,\|u-v\|\,\|w\| = C(M)\,\|u-v\|\,\|w\|$$

Where $(M) = (\epsilon\,\delta_1\, M^p + \delta_2)$. Hence, the hypothesis (H_2) is satisfied.

4. Numerical Result

Consider the GKdV equation:

$$u_t + \epsilon u^p u_x + \mu u_{xxx} = 0 \tag{17}$$

for different values $p = 1,2$.

It is well Known that this equation has the single solution[15]:

$$u(x,t) = \left[\frac{c(p+1)(p+2)}{2\epsilon}\operatorname{sech}^2[k(x - x_0 - ct)]\right]^{1/p} \tag{18}$$

where $k = \frac{p}{2}\sqrt{\frac{c}{\mu}}$.

When we put $p = 1$ we get the KdV equation, consider the boundary condition:

$$u(0,t) = g_1(t), u_x(0,t) = g_2(t), u_{xx}(1,t) = g_3(t) \tag{19}$$

Where $g_i(t)$ corresponds to the data from the exact solution.

By using ADM where $c = 1.3, \epsilon = 3, \mu = 1$ and $x_0 = 7.5\sqrt{c}$ we get

$$u_0(x,t) = \frac{1}{2}x^2 g_2(t) + x g_1(t) + g_1(t)$$

$$A_0 = u_0(x,t)[u_0(x,t)]_x \tag{20}$$

$$u_1(x,t) = \int_0^x \int_0^x \int_1^x (-3A_0 - [u_0(x,t)]_t)\, dx\, dx\, dx$$

$$A_1 = u_1(x,t)[u_0(x,t)]_x + u_0(x,t)[u_1(x,t)]_x$$

$$u_2(x,t) = \int_0^x \int_0^x \int_1^x (-3A_1 - [u_1(x,t)]_t)\, dx\, dx\, dx$$

and so on, then we have the approximation solution $u_{ap}(x,t)$:

$$u_{ap}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (21)$$

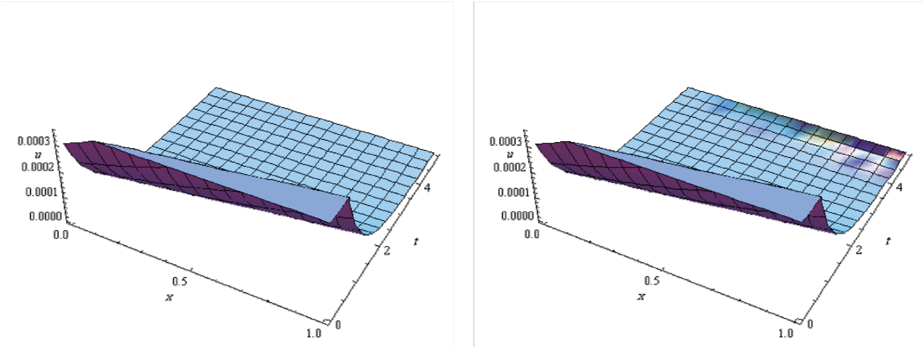
From Table 1, we find that ADM has a small error. The L_2 and L_∞ errors are contained in Table 2. In Fig. 1 the surface shows the exact solution (a) and the approximation solution (b) of KdV equation (GKdV, $p = 1$) for $0 \le t \le 5$, and $0 \le x \le 1$.

Table 1. Absolute error between the exact solution and approximation solution.

t/x	0.2	0.4	0.6	0.8	1
1	1.80521×10^{-7}	7.17809×10^{-7}	1.58906×10^{-6}	2.73578×10^{-6}	4.04845×10^{-6}
2	4.12294×10^{-8}	1.63941×10^{-7}	3.62929×10^{-7}	6.24832×10^{-7}	9.24639×10^{-7}
3	9.37623×10^{-9}	3.72828×10^{-8}	8.25358×10^{-8}	1.42097×10^{-7}	2.10278×10^{-7}
4	2.13039×10^{-9}	8.4711×10^{-9}	1.87531×10^{-8}	3.22859×10^{-8}	4.7777×10^{-8}
5	4.91693×10^{-10}	1.97594×10^{-9}	4.41198×10^{-9}	7.39243×10^{-9}	1.07838×10^{-8}

Table 2. L_2 and L_∞ errors.

t	L_2	L_∞
1	2.32152×10^{-6}	4.04845×10^{-6}
2	5.3022×10^{-7}	9.24639×10^{-7}
3	1.20581×10^{-7}	2.10278×10^{-7}
4	2.73971×10^{-8}	4.7777×10^{-8}
5	6.23778×10^{-9}	1.07838×10^{-8}



(a) The surface of the exact solution. **(b) The surface of the approximation solution.**

Fig. 1. The surface shows the exact and the approximation solution of KdV equation for $0 \leq t \leq 5$ and $0 \leq x \leq 1$.

When we put $p = 2$ we get the MKdV equation and by ADM where $c = 1, \epsilon = 6, \mu = 1$ and $x_0 = 0$ we get

$$u_0(x,t) = \frac{1}{2}x^2 g_2(t) + x g_1(t) + g_1(t)$$
$$A_0 = u_0(x,t)^2 [u_0(x,t)]_x \tag{22}$$

$$u_1(x,t) = \int_0^x \int_0^x \int_1^x (-6A_0 - [u_0(x,t)]_t)\, dx\, dx\, dx$$
$$A_1 = 2u_0(x,t)u_0(x,t)[u_0(x,t)]_x + u_0(x,t)^2[u_1(x,t)]_x$$
$$u_2(x,t) = \int_0^x \int_0^x \int_1^x (-6A_1 - [u_1(x,t)]_t)\, dx\, dx\, dx$$

and so on, then we have:

$$u_{ap}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{23}$$

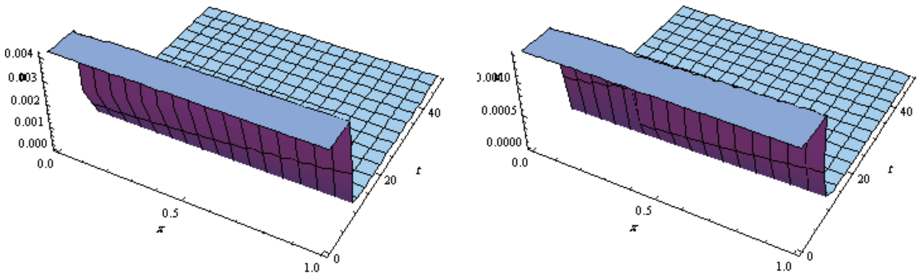
From Table 3, we find that ADM has a small error. The L_2 and L_∞ errors are contained in Table 4. In Fig. 2 the surface shows the exact solution (a) and the approximation solution (b) of MKdV equation (GKdV, $p = 2$) for $0 \leq t \leq 50$, and $0 \leq x \leq 1$.

Table 3. Absolute error between the exact solution and approximation solution.

t/x	0.2	0.4	0.6	0.8	1
10	7.9373×10^{-8}	2.85056×10^{-7}	6.17804×10^{-7}	1.05734×10^{-6}	1.56517×10^{-6}
20	3.04526×10^{-12}	1.21088×10^{-11}	2.68061×10^{-11}	4.61502×10^{-11}	6.82937×10^{-11}
30	1.38253×10^{-16}	5.49737×10^{-16}	1.21699×10^{-15}	2.09521×10^{-15}	3.10052×10^{-15}
40	6.27669×10^{-21}	2.4958×10^{-20}	5.52513×10^{-20}	9.51225×10^{-20}	1.40764×10^{-19}
50	2.84961×10^{-25}	1.13309×10^{-24}	2.50841×10^{-24}	4.31855×10^{-24}	6.39065×10^{-24}

Table 4. L_2 and L_∞ errors.

t	L_2	L_∞
10	8.98557×10^{-7}	1.56517×10^{-6}
20	3.9162×10^{-11}	6.82937×10^{-11}
30	1.77795×10^{-15}	3.10052×10^{-15}
40	8.07188×10^{-20}	1.40764×10^{-19}
50	3.66463×10^{-24}	6.39065×10^{-24}



(a) The surface of the exact solution. **(b) The surface of the approximation solution.**

Fig. 2. The surface shows the exact and the approximation solution of MKdV equation for $0 \le t \le 50$ and $0 \le x \le 1$.

5. Conclusions

In this paper, the ADM has been successful in finding the solution of KdV and MKdV equations with boundary conditions. A clear conclusion can be drawn from the numerical results that the ADM algorithm provides highly accurate numerical solutions without spatial discretizations for the nonlinear partial differential equations. In our work we use MATHEMATICA 7 for the direct evaluation of the Adomian polynomials of the obtained series solution.

Finally, it is worthwhile to mention that the ADM can be applied to other nonlinear partia differential equations in mathematical physics.

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حل معادلة كورتويج ديفرز المعممة ذات الشروط الحدية باستخدام طريقة أدومين للتجزئة

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المستخلص. في هذا البحث طبقت طريقة أدومين للتجزئة لحل معادلة كورتويج ديفرز المعممة ذات الشروط الحدية. وتقوم هذه الطريقة على تقديم الحل في صورة متسلسلة من كثيرات الحدود تتقارب إلى الحل المضبوط للمعادلة. وتمتاز طريقة أدومين للتجزئة عن غيرها من الطرق العددية السابقة بأنها تعطي مباشرة نظامًا للحل يتضمن الحد غير الخطي مما يعطي دقة أكبر في النتائج. وباستخدام برنامج الماثيماتيكا تم حساب كثيرات حدود أدومين في متسلسلة الحل.