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κ-normality and products of ordinals

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Abstract

A regular topological space is called *κ-normal* if any two disjoint regular closed subsets can be separated. In this paper we will show that any product of ordinals is κ-normal. In addition a generalization of a theorem of van Douwen and Vaughan will be proven and used to give an alternate proof that the product of any countable family of ordinals is κ-normal.

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E.V. Shchepin introduced, in [10], the class of κ-normal (also called mildly normal) topological spaces. A regular topological space is called *κ-normal* if any two disjoint regular closed subsets can be separated. Recall that a subset A of a topological space X is said to be *regular closed* (also called κ-closed or canonically closed) if $A = \overline{\operatorname{int} A}$. A subset A is said to be *regular open* (or κ-open or canonically open) if $A = \operatorname{int}(\overline{A})$. Two subsets A and B of a space X are said to be *separated* if there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A subspace P of Q is *$C^\star$-embedded* if any bounded continuous real-valued function on P can be continuously extended to Q. If Y is a subspace of X, then X is *normal on* Y, see [1], if any pair A and B of closed disjoint subsets of X such that $A = \overline{A \cap Y}^X$ and $B = \overline{B \cap Y}^X$ can be separated. X is *densely normal* if there is a dense subspace Y of X such that X is normal on Y. Any densely normal space is κ-normal [1], but not every κ-normal space is densely normal [4].

In [6], the class of κ-normal spaces was further studied. It was shown that most pathologies present for normal spaces also appear for κ-normality. Also, many standard

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non-normal spaces are κ-normal. For example, the square of the Sorgenfrey line, ω^{ω_1}, $\omega_1 \times (\omega_1 + 1)$, and the Tychonoff plank are κ-normal but not normal.

In this paper we show that any product of ordinals is κ-normal. The first section contains a proof of the full result. An alternate proof for the countable case will be given in the second section. In fact, we prove a stronger result for the countable case, that the product of any countable family of ordinals is densely normal. Towards proving this result, an extension of a theorem of van Douwen and Vaughan will be established.

The following notation will be used: For any subset $K \subseteq J$, let $\pi_K : \prod_{j\in J} X_j \to \prod_{j\in K} X_j$ be the natural projection. For any $i \in J$ and any subset $U \subseteq \prod_{j\in J} X_j$, let $U_i = \pi_{\{i\}} U$. For a point $x \in \prod_{j\in J} X_j$, let $x(i)$ denote the ith coordinate of x. For a basic open subset $U \subseteq \prod_{j\in J} X_j$, let $\operatorname{supp} U = \{i \in J\colon U_i \neq X_i\}$. If A is a set, then $[A]^{<\omega}$ denotes the set of all finite subsets of A and $[A]^{\leqslant\omega}$ denotes the set of countable subsets of A. Elementary submodels are used extensively in the first section. See [5] for the necessary background and notation on elementary submodel techniques.

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1. Arbitrary products of ordinals are κ-normal

This section is devoted to the proof of the following theorem:

Theorem 1. *If α_i is an ordinal for each $i < \lambda$, then $Z = \prod_{i<\lambda} \alpha_i$ is κ-normal.*

Proof. Let A and B be any nonempty regular closed disjoint subsets of Z. Write $A = \overline{\bigcup \mathcal{U}}$ and $B = \overline{\bigcup \mathcal{V}}$, where $\mathcal{U}$ and $\mathcal{V}$ are collections of basic open subsets of Z. Choose a sufficiently large θ, and let $\mathcal{M} \prec H_\theta$ be a countable elementary submodel such that A, B, $\mathcal{U}$, $\mathcal{V}$, λ, and $\langle \alpha_i\colon i < \lambda\rangle \in \mathcal{M}$. For each $i \in \mathcal{M} \cap \lambda$, we have $X_i^* = \mathcal{M} \cap \alpha_i$ is of the form $\bigcup_{j\in J_i} [\beta_j, \beta_{j+1})$, a pairwise disjoint union. Give each X_i^* the order topology. Note that this topology is in general coarser than the subspace topology. For example, if $\alpha_i \geqslant \omega_1$, then $\omega_1 \in X_i^*$ and ω_1 is a limit point of $\mathcal{M} \cap \omega_1$. Now, for each $i \in \mathcal{M} \cap \lambda$, there is $z_i \in \omega_1$ such that X_i^*, with the order topology, is homeomorphic to z_i. For each $i \in \mathcal{M} \cap \lambda$, define X_i as follows. If X_i^* is unbounded in α_i, let $X_i = X_i^*$. In the case that X_i^* is bounded in α_i, let $X_i = X_i^* \cup \{\sup X_i^*\}$; so, X_i is the one-point compactification of X_i^*. Finally, let $X = \prod_{i\in \mathcal{M}\cap\lambda} X_i$. Now, for each $U \in \mathcal{U} \cap M$, let $U^* = \pi_{(\mathcal{M}\cap\lambda)} U$, and let $U' = U^* \cap X$. For each $V \in \mathcal{V} \cap M$, define V^* and V' in a similar way. Let

$$A' = \overline{\bigcup\{U'\colon U \in \mathcal{U} \cap M\}}^X \quad \text{and} \quad B' = \overline{\bigcup\{V'\colon V \in \mathcal{V} \cap M\}}^X.$$

Claim 1. $A' \cap B' = \emptyset$.

Proof of Claim 1. Suppose not. Pick $x \in A' \cap B'$. For each $i \in \mathcal{M} \cap \lambda$, let $a_i = \sup(\mathcal{M} \cap x(i))$. Note that if $\mathcal{M} \cap x(i)$ is unbounded in $x(i)$, then $a_i = x(i)$, and if $\mathcal{M} \cap x(i)$ is bounded in $x(i)$, then $a_i < x(i)$. Let $y \in Z$ be such that $y(i) = a_i$ for each $i \in \mathcal{M} \cap \lambda$. Let

W be any basic open neighborhood of y in Z. For each $i \in \operatorname{supp} W \cap \mathcal{M}$, let $W_i = (\beta_i, y(i)]$ and without loss of generality, we may assume that $\beta_i \in X_i$ for each $i \in \operatorname{supp} W \cap \mathcal{M}$. Note that for each $i \in \operatorname{supp} W \cap \mathcal{M}$, we have that $\beta_i < y(i) \leqslant x(i)$ and hence $(\beta_i, x(i)] \cap X_i$ is a neighborhood of $x(i)$ in X_i. Define $W' \subset X$ as follows: For each $i \in \mathcal{M} \cap \lambda$, put

$$W'_i = \begin{cases} X_i, & \text{if } i \notin \operatorname{supp} W \cap \mathcal{M}, \\ (\beta_i, x(i)] \cap X_i, & \text{if } i \in \operatorname{supp} W \cap \mathcal{M}. \end{cases}$$

Then W' is an open neighborhood of x in X. Thus there exists $U \in \mathcal{U} \cap M$ and $V \in \mathcal{V} \cap M$ such that for each $i \in \operatorname{supp} W \cap \mathcal{M}$ we have that

$$\big((\beta_i, x(i)] \cap X_i\big) \cap U_i \neq \emptyset \neq \big((\beta_i, x(i)] \cap X_i\big) \cap V_i.$$

Let $i \in \operatorname{supp} U \cap \operatorname{supp} W \subseteq \operatorname{supp} W \cap \mathcal{M}$. Then we always have that $(\beta_i, x(i)] \cap X_i \subseteq (\beta_i, y(i)]$, thus W_i meets U_i. Thus $W \cap U \neq \emptyset$. Similarly, $W \cap V \neq \emptyset$. Thus $y \in A \cap B$, a contradiction. Thus Claim 1 is proved.

Now, for each $x \in Z$, define $x' \in X$ as follows: For each $i \in \mathcal{M} \cap \lambda$, put

$$x'(i) = \begin{cases} x(i), & \text{if } x(i) \in \mathcal{M}, \\ \min((\mathcal{M} \cap \alpha_i) \setminus x(i)), & \text{if } x(i) \notin \mathcal{M} \text{ and there is such a minimum}, \\ \sup(\mathcal{M} \cap \alpha_i), & \text{otherwise}. \end{cases}$$

Note that if $x \in Z$ and $i \in \mathcal{M} \cap \lambda$ such that $x(i) \notin \mathcal{M}$ and $\min((\mathcal{M} \cap \alpha_i) \setminus x(i)) = x'(i) \in \mathcal{M}$, then $x(i) < x'(i)$, and if $x(i) \notin \mathcal{M}$ and $\sup(\mathcal{M} \cap \alpha_i) = x'(i) \in \mathcal{M}$, then $x'(i) < x(i)$.

Claim 2. *If $x \in A$, then $x' \in A'$, and if $x \in B$, then $x' \in B'$.*

Proof of Claim 2. Let $x \in A$ be arbitrary. Let W' be an arbitrary open neighborhood of x' in X. We need to show that there exists $U \in \mathcal{U} \cap M$ such that $U' \cap W' \neq \emptyset$. Note that for each $i \in \operatorname{supp} W' = F$ there exists $\beta_i \in X_i$ such that $\beta_i < x'(i)$ and $W'_i = (\beta_i, x'(i)] \cap X_i$. By the definition of x' we have that for each $i \in F$, $\beta_i < x(i)$. Let $G = \{i \in F\colon x(i) \leqslant x'(i)\}$; and $K = \{i \in F\colon x(i) > x'(i)\}$. Define $W \subset Z$ as follows: For each $i < \lambda$, put

$$W_i = \begin{cases} \alpha_i, & \text{if } i \notin F, \\ (\beta_i, x(i)], & \text{if } i \in F. \end{cases}$$

Then W is an open neighborhood of x in Z. Thus there exists $U^0 \in \mathcal{U}$ such that $U^0 \cap W \neq \emptyset$, which means that the following statement Φ is true:

Φ: There exists $U^0 \in \mathcal{U}$ such that for each $i \in \operatorname{supp} U^0 \cap \operatorname{supp} W \subseteq F$ we have $U^0_i \cap (\beta_i, x(i)] \neq \emptyset$.

Since $\mathcal{U}$, F, (β_i, α_i), $\operatorname{supp} U^0 \cap F$ and β_i for each $i \in F$ are all in $\mathcal{M}$, then by elementarity of $\mathcal{M}$ we conclude that there exists $U \in \mathcal{U} \cap \mathcal{M}$ such that for each $i \in \operatorname{supp} U \cap \operatorname{supp} W \subseteq F$ we have that if $i \in G$, then $(U_i \cap (\beta_i, x(i)]) \cap \mathcal{M} \neq \emptyset$. And if $i \in K$, then $(U_i \cap (\beta_i, \alpha_i)) \cap \mathcal{M} \neq \emptyset$. This can be done even though $x(i)$ may not be an element of $\mathcal{M}$ (indeed, replace $(\beta_i, x(i)]$ by $(\beta_i, x'(i)]$ or by (β_i, α_i) depending on which case $x'(i)$ was defined). Now pick such a $U \in \mathcal{U} \cap \mathcal{M}$ and let $i \in \operatorname{supp} U \cap \operatorname{supp} W \subseteq F$ be arbitrary.

Observe that if $i \in G$, then $\emptyset \neq (U_i \cap (\beta_i, x(i)]) \cap \mathcal{M} = U_i' \cap (\beta_i, x'(i)]$; and if $i \in K$, then $\emptyset \neq (U_i \cap (\beta_i, \alpha_i)) \cap \mathcal{M} = U_i' \cap (\beta_i, x'(i)]$. Thus we have found $U \in \mathcal{U} \cap \mathcal{M}$ such that $U' \cap W' \neq \emptyset$, hence $x' \in A'$. Similar argument will show that if $x \in B$, then $x' \in B'$. So Claim 2 is proved.

Now, A' and B' are regular closed disjoint in $X = \prod_{i \in \mathcal{M} \cap \lambda} X_i$. Since $|\mathcal{M} \cap \lambda| \leqslant \aleph_0$ and $X_i \cong z_i \in \omega_1$ for each $i \in \mathcal{M} \cap \lambda$, then X is metrizable. So, fix open disjoint subsets G and H of X such that $A' \subseteq G$ and $B' \subseteq H$. For each $x \in A$, fix a basic open neighborhood $U(x')$ of x' in X such that $U(x') \subseteq G$. Note that for each $i \in \operatorname{supp} U(x')$ there exists $\beta_i < x'(i)$ such that $\beta_i \in X_i$ and $U(x')_i = (\beta_i, x'(i)] \cap X_i$ and by the definition of x' we always have that $\beta_i < x(i)$. Define an open neighborhood $U(x)$ of x in $Z = \prod_{i<\lambda} \alpha_i$ as follows: For each $i < \lambda$, put

$$U(x)_i = \begin{cases} \alpha_i, & \text{if } i \notin \operatorname{supp} U(x'), \\ (\beta_i, x(i)], & \text{if } i \in \operatorname{supp} U(x') \text{ and } x(i) \leqslant x'(i), \\ (x'(i), x(i)], & \text{if } i \in \operatorname{supp} U(x') \text{ and } x'(i) < x(i). \end{cases}$$

Similarly, for each $y \in B$, fix a basic open neighborhood $V(y')$ of y' in X such that $V(y') \subseteq H$. Note that for each $i \in \operatorname{supp} V(y')$ there exists $\gamma_i < y'(i)$ such that $\gamma_i \in X_i$ and $V(y')_i = (\gamma_i, y'(i)] \cap X_i$ and by the definition of y' we always have that $\gamma_i < y(i)$. Define an open neighborhood $V(y)$ of y in Z as follows: for each $i < \lambda$, put

$$V(y)_i = \begin{cases} \alpha_i, & \text{if } i \notin \operatorname{supp} V(y'), \\ (\gamma_i, y(i)], & \text{if } i \in \operatorname{supp} V(y') \text{ and } y(i) \leqslant y'(i), \\ (y'(i), y(i)], & \text{if } i \in \operatorname{supp} V(y') \text{ and } y'(i) < y(i). \end{cases}$$

Claim 3. *$U(x) \cap V(y) = \emptyset$ for each $x \in A$ and $y \in B$.*

Proof of Claim 3. Suppose not, then there exists $x \in A$ and $y \in B$ such that $U(x) \cap V(y) \neq \emptyset$. Since $U(x') \cap V(y') = \emptyset$, then there is an $i \in \operatorname{supp} U(x) \cap \operatorname{supp} V(y)$ which satisfy $U(x')_i \cap V(y')_i = \emptyset$. This implies that either $\beta_i < x'(i) \leqslant \gamma_i < y'(i)$ or $\gamma_i < y'(i) \leqslant \beta_i < x'(i)$.

Case 1. $x(i) \leqslant x'(i)$ and $y(i) \leqslant y'(i)$. So, $U(x)_i = (\beta_i, x(i)] \subseteq (\beta_i, x'(i)]$ and $V(y)_i = (\gamma_i, y(i)] \subseteq (\gamma_i, y'(i)]$. If $x'(i) = y'(i)$, then $U(x')_i \cap V(y')_i \neq \emptyset$, a contradiction. So, assume, without loss of generality, $x'(i) < y'(i)$. Since $(\beta_i, x'(i)] \cap (\gamma_i, y'(i)] \cap X_i = \emptyset$ and $\gamma_i \in \mathcal{M}$, then $x'(i) \leqslant \gamma_i$. Thus $(\beta_i, x(i)] \cap (\gamma_i, y(i)] = U(x)_i \cap V(y)_i = \emptyset$, a contradiction.

Case 2. $x(i) \leqslant x'(i)$ and $y'(i) < y(i)$. This means that $x'(i) < \sup(\mathcal{M} \cap \alpha_i) = y'(i)$, so $U(x)_i \cap V(y)_i = \emptyset$, a contradiction.

Case 3. $x'(i) < x(i)$ and $y(i) \leqslant y'(i)$. This case is similar to case 2.

Case 4. $x'(i) < x(i)$ and $y'(i) < y(i)$. This means $x'(i) = \sup(\mathcal{M} \cap \alpha_i) = y'(i)$, thus $U(x')_i \cap V(y')_i \neq \emptyset$, a contradiction.

So, in all cases we get a contradiction, so Claim 3 is proved.

Define $U(A) = \bigcup_{x \in A} U(x)$ and $V(B) = \bigcup_{y \in B} V(y)$, then $U(A)$ and $V(B)$ are open in Z containing A and B, respectively. By Claim 3, we conclude that $U(A) \cap V(B) = \emptyset$. So, A and B can be separated, hence Z is κ-normal. This completes the proof of Theorem 1. □

2. Countable products of ordinals are densely normal

In this section we will give an alternate proof for the countable case. It will be a corollary for the following theorem

Theorem 2. *Suppose that α_i is an ordinal for each $i \in \omega$. Then $\prod\{\alpha_i\colon i \in \omega\}$ is densely normal.*

To prove Theorem 2 we will prove a theorem on normality of products of certain subspaces of ordinals. This result extends a theorem of van Douwen and Vaughan.

In [7] (see also [8]), Nogura defined for an infinite cardinal τ and an ordinal α, the subspace $S(\tau, \alpha)$ of the ordinal space $\alpha + 1$ by

$$S(\tau, \alpha) = \{\beta \leqslant \alpha\colon \operatorname{cf}(\beta) \leqslant \tau\}.$$

He proved the following:

Theorem 3 (Nogura). *If τ is an infinite cardinal, then $(S(\tau, \alpha))^\omega$ is normal for any ordinal α.*

In [2], van Douwen and Vaughan gave a generalization of Theorem 3. They defined for each uncountable cardinal τ and each infinite ordinal α, the subspace $S'(\tau, \alpha)$ of the ordinal space $\alpha + 1$:

$$S'(\tau, \alpha) = \{\beta \leqslant \alpha\colon \operatorname{cf}(\beta) < \tau\}.$$

They proved the following:

Theorem 4 (van Douwen and Vaughan). *If τ is uncountable, $\lambda < \tau$, and α_i are infinite ordinals for each $i < \lambda$, then $\prod\{S'(\tau, \alpha_i)\colon i < \lambda\}$ is normal.*

Also, they gave the following corollary to their theorem:

Corollary 1 (van Douwen and Vaughan). *If τ is infinite and $\lambda \leqslant \tau$, then $\prod_{i<\lambda} S(\tau, \alpha_i)$ is normal.*

Now, let τ be an uncountable cardinal and α be any ordinal. Define the subspace $S''(\tau, \alpha)$ of the ordinal space α by

$$S''(\tau, \alpha) = \{\beta < \alpha\colon \operatorname{cf}(\beta) < \tau\}.$$

The version of Theorem 4 for S'' is false whenever $\tau > \omega_1$. Indeed, if $\omega < \lambda < \tau$ then one need only consider the non-normal product ω^λ and if $2 \leqslant \lambda \leqslant \omega$ it suffices to consider $\omega_1 \times (\omega_1 + 1)$. However, if $\tau = \omega_1$ then we obtain the following theorem not covered by the theorems of Nogura or van Douwen and Vaughan.

Theorem 5. *If α_i is an ordinal for each $i < \omega$, then $\prod\{S''(\omega_1, \alpha_i)\colon i < \omega\}$ is normal.*

Proof. Fix $\langle \alpha_i : i \in \omega \rangle$. To simplify our notation, let $Y_i = S''(\omega_1, \alpha_i)$ for each $i < \omega$, and $Y = \prod_{i \in \omega} Y_i$. Then Y is first countable being a countable product of first countable spaces. The following theorem from [13] will be used:

Theorem 6 (Zenor). *Suppose that all finite subproduct of a product space $Z = \prod_{i<\omega} Z_i$ are normal, then Z is normal if and only if Z is countably paracompact.*

Also, we need the following lemma whose straightforward proof we leave to the reader.

Lemma 1. *If for each $i \in \omega$ either $\mathrm{cf}(\alpha_i) > \omega$ or $\mathrm{cf}(\alpha_i) = 1$, then Y is countably compact.*

To complete the proof we will show that any finite subproduct of Y is normal and that Y is countably paracompact. Applying Zenor's theorem will complete the proof.

First consider the case that for each $i \in \omega$, α_i is infinite and either $\mathrm{cf}(\alpha_i) > \omega$ or $\mathrm{cf}(\alpha_i) = 1$. Partition ω into two subsets A and B such that $\mathrm{cf}(\alpha_i) > \omega$ for each $i \in A$ and $\alpha_i = \zeta_i + 1$ for each $i \in B$. Note that for each $i \in A$ we have

$$Y_i = \{\beta < \alpha_i : \mathrm{cf}(\beta) < \omega_1\} = \{\beta < \alpha_i + 1 : \mathrm{cf}(\beta) < \omega_1\} = S'(\omega_1, \alpha_i),$$

and for each $i \in B$ we have

$$Y_i = \{\beta < \alpha_i : \mathrm{cf}(\beta) < \omega_1\} = \{\beta \leqslant \zeta_i : \mathrm{cf}(\beta) < \omega_1\} = S'(\omega_1, \zeta_i).$$

Therefore, by Theorem 4, $\prod_{i \in \omega} Y_i = Y$ is normal. Second, assume that for each $i \in \omega$ either $\mathrm{cf}(\alpha_i) > \omega$ or $\mathrm{cf}(\alpha_i) = 1$ but there are some $i \in \omega$ such that α_i is finite. Partition $\omega = E \cup F$ where α_i is infinite for each $i \in E$ and α_i is finite for each $i \in F$. Then for each $i \in F$, $Y_i = \alpha_i$ which is compact, hence $\prod_{i \in F} Y_i$ is T_2-compact metrizable and $\prod_{i \in E} Y_i$ is countably compact (by Lemma 1) and normal. Thus by Stone's theorem, see [12], we get that $Y = (\prod_{i \in F} Y_i) \times (\prod_{i \in E} Y_i)$ is normal.

Claim 4. *For each $n \in \omega$, $\prod_{i \leqslant n} Y_i$ is normal. (Hence any finite subproduct of Y is normal.)*

Let $A = \{\alpha_i : \mathrm{cf}(\alpha_i) \neq \omega\}$ and $B = \{\alpha_i : \mathrm{cf}(\alpha_i) = \omega\}$. If $B = \emptyset$ then the product is normal as above, and if $B \neq \emptyset$ then the product can be written as a direct sum of clopen normal subspaces.

Claim 5. *Y is countably paracompact.*

Proof of Claim 5. The proof of this claim is rather tedious but straightforward.

If for each $i \in \omega$, $\mathrm{cf}(\alpha_i) > \omega$ or $\mathrm{cf}(\alpha_i) = 1$, then we have by Lemma 1 that Y is countably compact, hence countably paracompact. So write $\omega = A \cup B$, where $\mathrm{cf}(\alpha_i) > \omega$ or $\mathrm{cf}(\alpha_i) = 1$ for each $i \in A$ and $\mathrm{cf}(\alpha_i) = \omega$ for each $i \in B$. And assume $B \neq \emptyset$. If B is finite, then Y can be written as a direct sum of clopen countably paracompact subspaces of Y, thus Y is countably paracompact.

So, assume now B is infinite. For each $i \in B$, define $L_{\alpha_i} = \{\beta < \alpha_i : \mathrm{cf}(\beta) > \omega\}$. And let $\alpha_i^* = \sup(L_{\alpha_i})$. We are going to define for each $i \in B$ a countable ordinal $z_i < \omega_1$ and a continuous open and onto function $f_i : \alpha_i \to z_i$ by considering the following possible cases.

Case 1. $L_{\alpha_i} = \emptyset$, then $\alpha_i < \omega_1$: Let $z_i = \alpha_i$ and let $f_i =$ the identity map.

Case 2. $\alpha_i = \alpha_i^\star$: Choose $\beta_i^n \in L_{\alpha_i}$ increasing and cofinal in α_i such that $\beta_i^0 = 0$. Let $z_i = \omega$ and let f_i be defined so that $f_i^{-1}(n) = (\beta_i^n, \beta_i^{n+1}]$.

Case 3. $\alpha_i^\star = \max L_{\alpha_i} < \alpha_i$: Let $z_i = \omega$ and choose $\{\beta_i^n \mid n \in \omega\}$ an increasing cofinal in α_i sequence of successor ordinals with $\beta_i^0 = 0$ and $\beta_i^n > \alpha_i^\star$ for $n > 0$ and define f_i as in Case 2.

Case 4. $\alpha_i^\star = \sup L_{\alpha_i}, \alpha_i^\star < \alpha_i$ and $\mathrm{cf}(\alpha_i^\star) = \omega$: Let $z_i = \omega + \omega$ and choose $\{\beta_i^n \mid n \in \omega + \omega\}$ an increasing cofinal in α_i sequence of ordinals such that $\{\beta_i^n\colon n \in \omega\}$ is as in Case 2, and $\{\beta_i^n \mid \omega \leqslant n < \omega + \omega\}$ is as in Case 3 and define f_i as in Case 2.

For each $i \in B$, let $g_i = f_i|Y_i$, the restriction of f_i to Y_i. Define $g : (\prod_{i \in A} Y_i) \times (\prod_{i \in B} Y_i) \to (\prod_{i \in A} Y_i) \times (\prod_{i \in B} z_i)$ by $g = \prod_{i \in \omega} g_i$. It can be shown that

(I) For each $y \in (\prod_{i \in A} Y_i) \times (\prod_{i \in B} z_i)$, $g^{-1}\{y\}$ is a countably compact subset of $(\prod_{i \in A} Y_i) \times (\prod_{i \in B} Y_i) = Y$.

(II) g is a closed mapping.

So, the countable paracompactness of Y follows from Hanai's theorem, [3, Exercise 5.2.G]. This completes the proof of the claim.

Now, by Zenor's theorem, we may conclude that Y is normal. This completes the proof of Theorem 5. □

We now turn to the proof of Theorem 2. Let α_i be an ordinal for each $i \in \omega$ and let $X = \prod_{i \in \omega} \alpha_i$. For each $i \in \omega$, define $Y_i = \{\beta < \alpha_i\colon \mathrm{cf}(\beta) < \omega_1\} = S''(\omega_1, \alpha_i) \subseteq \alpha_i$, and let $Y = \prod_{i \in \omega} Y_i \subseteq X$. We will use the following theorem of Arhangel'skii, see [1]:

Theorem 7 (Arhangel'skii). *If P is a normal subspace of Q such that P is $C^\star$-embedded in Q, then Q is normal on P.*

Thus, to prove Theorem 2 it suffices to prove the following lemma:

Lemma 2. *Y is $C^\star$-embedded in X.*

Proof. By Taimonov's theorem [3, Theorem 3.2.1] it suffices to show that if E and F are any closed disjoint subsets of Y then $\overline{E} \cap \overline{F} = \emptyset$. By way of contradiction fix E and F closed subsets of Y and $x = \langle x_n\colon n \in \omega\rangle \in X$ such that $x \in \overline{E} \cap \overline{F}$. Partition $\omega = A \cup B$ such that $\mathrm{cf}(x_n) > \omega$ if and only if $n \in B$. Since $x \notin Y$ we have $B \neq \emptyset$. We consider only the case where B is infinite (the finite case is easier).

Enumerate B as $\{n_i\colon i \in \omega\}$. Let $\{U_n\colon n \in \omega\}$ be a local neighborhood base at $x|A$ in $\prod_{n \in A} \alpha_n$. We construct elements $a^m \in E$ and $b^m \in F$ recursively as follows. Let

$$W_0 = (0, x_{n_0}] \times \left(\prod_{n \in B \setminus \{n_0\}} \alpha_n\right) \times U_0.$$

W_0 is an open neighborhood of $x \in X$ so we may pick $a^0 \in E \cap W_0$. Let

$$V_0 = (a^0_{n_0}, x_{n_0}] \times \Big(\prod_{n \in B \setminus \{n_0\}} \alpha_n\Big) \times U_0.$$

V_0 is an open neighborhood of $x \in X$ so we may pick $b^0 \in F \cap W_0$.

Having chosen a^i and b^i for all $i < m$ let

$$W_m = \prod_{i<m} (b^i_{n_i}, x_{n_0}] \times \Big(\prod_{n \in B \setminus \{n_i:\ i<m\}} \alpha_n\Big) \times U_m.$$

W_m is an open neighborhood of $x \in X$ so we may pick $a^m \in E \cap W_m$. Let

$$V_m = \prod_{i<m} (a^i_{n_i}, x_{n_i}] \times \Big(\prod_{n \in B \setminus \{n_i:\ i<m\}} \alpha_n\Big) \times U_m.$$

V_m is an open neighborhood of $x \in X$ so we may pick $b^m \in F \cap W_0$.

For each $i \in \omega$ let $y_{n_i} = \sup\{a^m_{n_i}\colon m > i\}$ by construction it follows that also $y_{n_i} = \sup\{b^m_{n_i}\colon m > i\}$. In particular both sequences $\langle a^m | B\colon m \in \omega\rangle$ and $\langle b^m | B\colon m \in \omega\rangle$ converge to $y|B$.

For $n \in A$, let $y_n = x_n$. This defines $y \in Y$. To finish the proof we will reach a contradiction by showing that $y \in \overline{E} \cap \overline{F}$. Fix O a basic open neighborhood of y. So $O = G \times H$ where G is open in $\prod_{n\in B} \alpha_n$ and H is open in $\prod_{n \in A} \alpha_n$. Fix m large enough so that $U_m \subseteq H$ and such that both $a^m|B \in G$ and $b^m|B \in G$. Then since both $a^m|A \in U_m$ and $b^m|A \in U_m$ we have that both $a^m \in O$ and $b^m \in O$. This completes the proof. □

Since dense normality implies κ-normality, see [1], we obtain an alternate proof of the countable instance of Theorem 1:

Corollary 2. *Any countable product of ordinals is κ-normal.*

We conclude with the following natural problems:

Problem 1. *Is the product of any family of subspaces of ordinals κ-normal?*

Problem 2. *Let $X = \prod_{i \in I} X_i$. Is X κ-normal assuming either*
(a) *$\prod_{i\in J} X_i$ is κ-normal for each $J \in [I]^{<\omega}$; or*
(b) *$\prod_{i\in J} X_i$ is κ-normal for each $J \in [I]^{\leqslant\omega}$?*

The analogous problem for normal spaces has many interesting counterexamples (see [9]). In fact, we do not know whether any of these examples are κ-normal. So the problems are open even if we assume that the subproducts are, for example, normal or even Lindelöf. We do have a positive answer to the above problems in some special cases: Shchepin proved that the product of any family of κ-metrizable spaces is κ-metrizable (hence κ-normal), see [11], so no counterexample can consist of κ-metrizable spaces X_i.

If X is ccc and every countable subproduct is κ-normal, then X is κ-normal. This is because the closure of any open subset of X depends on only countably many coordinates (see [3]).

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