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Tightness of probability measures on function spaces

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ABSTRACT

Let $C_E = C([0\ 1], E)$ be the Banach space, with the supremum norm, of all continuous functions f from the unit interval $[0\ 1]$ into the Banach space E. If $E = \mathbb{R}$ we put $C_{\mathbb{R}} = C$. Function spaces under consideration are equipped with their Borel σ-field. This paper deals with the tightness property of some classes of probability measures (p.m) on the function space C_E. We will be concerned mainly with the specific cases $E = \mathbb{R}$, $E = C$ and more generally E a separable Banach space. We give sufficient conditions for tightness by extending and strengthening the conditions developed by Prohorov in connection with limit theorems of stochastic processes. In the general case of a separable Banach space E, the property of tightness will be settled under conditions of different nature from those of Prohorov. Finally weak convergence of p.m on C_E will be established under the condition of weak convergence of their finite dimensional distributions. This extends a similar result valid in the space C.

1. Introduction

Let S be a metric space with its Borel σ-field $\mathcal{B}_S$. Consider the set $\mathcal{P}(S)$ of all probability measures on $S, \mathcal{B}_S$ and equip $\mathcal{P}(S)$ with the weak* topology. As customary, we call the convergence in this topology the weak convergence of probability measures, defined as follows: A sequence (P_n) of p.m in $\mathcal{P}(S)$ is said to converge weakly to the p.m $P \in \mathcal{P}(S)$ if:

$$\int_S f\,dP_n \to \int_S f\,dP \tag{1.1}$$

for every bounded continuous function $f : S \to \mathbb{R}$ (symbol: $P_n \Rightarrow P$).

Also, for weak compactness in $\mathcal{P}(S)$, we shall adhere to the following definition:

A subset Γ of $\mathcal{P}(S)$ is called sequentially relatively compact (shortly relatively compact) if every sequence (P_n) in Γ contains a weakly convergent subsequence, that is, a subsequence $P_{n'}$ such that there is a p.m Q with $P_n \Rightarrow Q$.

The theory of weak convergence of probability measures has been extensively studied by several authors, especially for those considerations pertaining to limit theorems in probability theory, see [3,5–8]; a very nice account with more references, is in [1].

One of the most prominent feature of the theory is given by the theorem of Prohorov [6] who characterized the relative compactness in $\mathcal{P}(S)$ in terms of the so-called property of tightness:

Definition 1.1. A family Γ of probability measures in $\mathcal{P}(S)$ is said to be tight if, given any $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon$ of S such that:

$$P(K) > 1 - \varepsilon, \quad \text{for all } P \text{ in } \Gamma.$$

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The following theorem is due to Prohorov:

Theorem 1.2. *(See [6].) (a) Every tight family $\Gamma \subset \mathcal{P}(S)$ is relatively compact.*
(b) If S is a polish space, then every relatively compact family $\Gamma \subset \mathcal{P}(S)$ is tight.

See Ref. [1] for a detailed proof.

Part (a) of the theorem is more interesting for the applications. So it is important to have at one's disposal criteria for tightness. The following theorem gives a well-known criterion developed by Prohorov for families of p.m on the space C. First we need a definition and a lemma. In what follows, functions in C_E are denoted by capitals $X, Y, \ldots$ and those in C by small letters $x, y, \ldots$. The value of the function $X \in C_E$ for $t \in [0\ 1]$ is denoted by X_t, but for the functions $x \in C$ we prefer the notation $x(t)$.

Definition 1.3. The modulus of a function $X \in C_E$ is defined by:

$$\omega_X(\delta) = \operatorname*{Sup}_{|t-s|<\delta} \|X_t - X_s\| \tag{1.4}$$

where δ is a real number, usually with $0 < \delta < 1$. Similarly the modulus of a function $x \in C$ is defined by $\omega_x(\delta) = \operatorname{Sup}_{|t-s|<\delta} |x(t) - x(s)|$.

The proof of the following lemma is a routine job.

Lemma 1.5. (a) $\operatorname{Lim}_{\delta \to 0} \omega_X(\delta) = 0$, *for each $X \in C_E$. Moreover the limit is uniform on compact sets of C_E.*
(b) For every X, Y in C_E we have: $|\omega_X(\delta) - \omega_Y(\delta)| \leqslant 2\|X - Y\|$, for each $\delta > 0$. So the function $X \to \omega_X(\delta)$ is continuous and then Borel measurable.

Theorem 1.6. *(See [6].) A family Γ of p.m on C is tight if and only if the following conditions hold:*

(i) For each positive η there exists an a such that

$$P\{x:\ |x(0)| > a\} \leqslant \eta, \quad \text{for all } P \in \Gamma. \tag{1.7}$$

(ii) For each positive ε and η, there exist a δ, with $0 < \delta < 1$ such that

$$P\{x:\ \omega_x(\delta) \geqslant \varepsilon\} \leqslant \eta, \quad \text{for all } P \in \Gamma. \tag{1.8}$$

It is the objective of this work to establish criteria for tightness on function spaces of type C_E. We start, in Section 2, with a special class of p.m on the space C, induced by a family of p.m on C_E via simple continuous transformations. Next, in Section 3, we prove tightness on the function space $C([0\ 1], C)$ by strengthening conditions (1.7) and (1.8) of Theorem 1.6. In Section 4, we consider probability measures on C_E for a separable Banach space E. We establish tightness under conditions different from those of Sections 2 and 3.

2. Tightness of probability images

2.1. Let E^* be the topological dual of E. For each $x^* \in E^*$, define the operator $V_{x^*} : C_E \to C$ by

$$X \in C_E, \quad V_{x^*}(X) = x^* \circ X$$

where $x^* \circ X(s) = x^*(X_s)$, for $s \in [0\ 1]$.

It is easy to see that V_{x^*} is linear and bounded.

Now consider a family Γ of p.m on the space C_E and form the image $\Gamma \circ V_{x^*}^{-1}$ of Γ under V_{x^*}, that is the family of p.m on C given by $\Gamma \circ V_{x^*}^{-1} = \{P V_{x^*}^{-1},\ P \in \Gamma\}$. The following theorem gives conditions on Γ under which the whole family $\{P V_{x^*}^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$ is tight, σ_1^* being the closed unit ball of E^*.

Theorem 2.2. *With the ingredients above, assume the following conditions are satisfied by Γ:*

(a) For each positive η there exists an a such that

$$P\{X \in C_E:\ \|X_0\| > a\} \leqslant \eta, \quad \text{for all } P \in \Gamma. \tag{2.3}$$

(b) For each positive ε and η, there exist a δ, with $0 < \delta < 1$ such that

$$P\{X \in C_E:\ \omega_X(\delta) \geqslant \varepsilon\} \leqslant \eta, \quad \text{for all } P \in \Gamma. \tag{2.4}$$

Then the family $\{P V_{x^*}^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$ is tight on the space C.

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Proof. We show that (2.3) and (2.4) imply that conditions (1.7) and (1.8) are satisfied by the p.m $\{PV_{x^*}^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$. We have $V_{x^*}^{-1}\{x \in C\colon |x(0)| > a\} = \{X \in C_E\colon |x^*(X_0)| > a\}$ and since $|x^*(X_0)| \leqslant \|X_0\|$, for $x^* \in \sigma_1^*$, we deduce that $\{X \in C_E\colon |x^*(X_0)| > a\} \subset \{X \in C_E\colon \|X_0\| > a\}$. Consequently we have $PV_{x^*}^{-1}\{x \in C\colon |x(0)| > a\} \leqslant P\{X \in C_E\colon \|X_0\| > a\}$. Hence if (2.3) is satisfied, (1.7) will be too for all the p.m $PV_{x^*}^{-1}$, for $P \in \Gamma$ and $x^* \in \sigma_1^*$. On the other hand $V_{x^*}^{-1}\{x \in C\colon \omega_x(\delta) \geqslant \varepsilon\} = \{X \in C_E\colon \omega_{x^* \circ X}(\delta) \geqslant \varepsilon\}$ and by (1.4), we have $\omega_{x^* \circ X}(\delta) \leqslant \omega_X(\delta)$, for $x^* \in \sigma_1^*$. So we deduce that $\{X \in C_E\colon \omega_{x^* \circ X}(\delta) \geqslant \varepsilon\} \subset \{X \in C_E\colon \omega_X(\delta) \geqslant \varepsilon\}$. It follows from (2.4) that (1.8) is satisfied by all the p.m $PV_{x^*}^{-1}$, for $P \in \Gamma$ and $x^* \in \sigma_1^*$. This proves the tightness of the family $\{PV_{x^*}^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$. □

Remark. It seems difficult to get the tightness of the family Γ itself, only from conditions (a) and (b). More should be imposed either on Γ or on the space E. This problem has been considered in [4], in a somewhat different setting, when E is the dual of a nuclear Frechet space.

In Section 3 we will consider the case $E = C$, then by strengthening the preceding conditions on Γ, we get the tightness of this family with the help of the Arzela–Ascoli theorem for vector-valued functions.

First let us note the following consequence of Theorem 2.2.

Theorem 2.3. *Let λ be a probability measure on $[0\ 1]$ and let us consider the bounded operator $T : C_E \to E$ given by the Bochner integral:*

$$X \in C_E, \quad TX = \int_{[0\ 1]} X_s\, d\lambda(s).$$

Let Γ be a family of probability measures on C_E, satisfying the conditions (a) *and* (b) *of Theorem 2.2. Then the family $\{P(x^* \circ T)^{-1}, P \in \Gamma, x^* \in \sigma_1^*\}$ is a tight family of p.m on $\mathbb{R}$. The same conclusion holds with λ any bounded real measure on $[0\ 1]$.*

Proof. Let φ be the bounded functional on C given by $\varphi(x) = \int_{[0\ 1]} x(s)\, d\lambda(s)$, $x \in C$. Then we have $\varphi(V_{x^*}(X)) = \varphi(x^* \circ X) = \int_{[0\ 1]} x^*(X_s)\, d\lambda(s)$. But by the Bochner integral properties, we get:

$$\int_{[0\ 1]} x^*(X_s)\, d\lambda(s) = x^*\biggl(\int_{[0\ 1]} X_s\, d\lambda(s)\biggr) = x^*(TX).$$

So we deduce that $\varphi \circ V_{x^*} = x^* \circ T$, for every $x^* \in E^*$. From Theorem 2.2, the family of p.m $\{PV_{x^*}^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$ is tight. Consequently, by appealing to Lemma 1 of [1, p. 38], the family $\{PV_{x^*}^{-1}\varphi^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$ is tight on $\mathbb{R}$. By the relation just proved, this last family is exactly the family $\{P(x^* \circ T)^{-1},\ P \in \Gamma, x^* \in \sigma_1^*\}$ we need. □

3. Tightness of probability measures on the space $C([0\ 1], C)$

In this section we prove a tightness theorem for p.m on the space $C([0\ 1], C)$, similar to Prohorov theorem on the space C [1, Theorem 8.2]. First we need:

3.1. For each $X \in C([0\ 1], C)$, define the new modulus of X by the recipe:

$$M_X(\delta) = \operatorname{Sup}_t \omega_{X_t}(\delta) \tag{3.2}$$

where $\omega_{X_t}(\delta) = \operatorname{Sup}_{|u-v|<\delta} |X_t(u) - X_t(v)|$ is given by Definition 1.3 for $X_t \in C$.

Just a little work is needed to prove that $M_X(\delta)$ has the following features of a usual modulus:

(a) $|M_X(\delta) - M_Y(\delta)| \leqslant 2\|X - Y\|$, for all X, Y in $C([0\ 1], C)$.
(b) $M_X(\delta) \to 0$, $\delta \to 0$.

A little inspection at (1.4) and (3.2) shows that the moduli $M_X(\delta)$ and $\omega_X(\delta)$ are the measures of two different kinds of oscillations for the function X; however both are needed in the following theorem.

Theorem 3.3. *Let Γ be a family of probability measures on $C([0\ 1], C)$ satisfying the conditions:*

$$\forall \eta > 0,\ \exists a \geqslant 0\colon\quad P\Bigl\{X\colon \operatorname{Sup}_t \bigl|X_t(0)\bigr| \leqslant a\Bigr\} > 1 - \eta, \quad \text{for all } P \in \Gamma, \tag{3.4}$$

$$\forall \varepsilon > 0,\ \forall \eta > 0,\ \exists 0 < \delta < 1\colon\quad P\bigl\{X\colon \omega_X(\delta) < \varepsilon\bigr\} > 1 - \eta, \quad \text{for all } P \in \Gamma, \tag{3.5}$$

$$\forall \varepsilon > 0,\ \forall \eta > 0,\ \exists 0 < \delta < 1\colon\quad P\bigl\{X\colon M_X(\delta) < \varepsilon\bigr\} > 1 - \eta, \quad \text{for all } P \in \Gamma. \tag{3.6}$$

Then the family Γ is tight.

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Conditions (3.5), (3.6) prevent most of the functions X to have too big oscillations.

Proof. Let $\varepsilon > 0$ and let j be an integer with $j \geqslant 1$. Then choose a and δ_j, $0 < \delta_j < 1$, so that, if $A = \{X\colon \operatorname{Sup}_t |X_t(0)| \leqslant a\}$, $B_j = \{X\colon \omega_X(\delta_j) < \frac{1}{j}\}$, $D_j = \{X\colon M_X(\delta_j) < \frac{1}{j}\}$, we have:

$$P(A) > 1 - \frac{\varepsilon}{2}, \qquad P(B_j) > 1 - \frac{\varepsilon}{2^{j+2}}, \qquad P(D_j) > 1 - \frac{\varepsilon}{2^{j+2}}, \quad \text{for all } P \in \Gamma.$$

Now put $B = \bigcap_j B_j$, $D = \bigcap_j D_j$ and $H = A \cap B \cap D$. Then we have for all $P \in \Gamma$, $P(H) > 1 - \varepsilon$. We show that H is relatively compact. By the Arzela–Ascoli theorem for vector-valued functions [2, p. 81], it is enough to prove that H is equicontinuous and that the orbit set $H_t = \{X_t,\ X \in H\}$ is relatively compact in C for each $t \in [0\ 1]$. Since $H \subset B_j$, $\forall j$, we have $\operatorname{Sup}_{X \in H} \omega_X(\delta_j) \leqslant \operatorname{Sup}_{X \in B} \omega_X(\delta_j) \leqslant \frac{1}{j}$, $\forall j$. So we deduce that $\operatorname{Lim}_j \operatorname{Sup}_{X \in H} \omega_X(\delta_j) = 0$. From this it follows that $\operatorname{Lim}_{\delta \to 0} \operatorname{Sup}_{X \in H} \omega_X(\delta) = 0$, whence the equicontinuity of H.

We turn to the relative compactness of the set H_t. We apply the Arzela–Ascoli theorem for scalar functions in the space C [1, p. 221].

First we have $\operatorname{Sup}_{X \in H} |X_t(0)| \leqslant \operatorname{Sup}_{X \in A} |X_t(0)| < \infty$, since $H \subset A$. On the other hand we have also $H \subset D$; so for every j, $\operatorname{Sup}_{X \in H} M_X(\delta_j) \leqslant \operatorname{Sup}_{X \in D} M_X(\delta_j) \leqslant \frac{1}{j}$.

Consequently $\operatorname{Lim}_j \operatorname{Sup}_{X \in H} \omega_{X_t}(\delta_j) = 0$, and then it follows that $\operatorname{Lim}_{\delta \to 0} \operatorname{Sup}_{X \in H} \omega_{X_t}(\delta) = 0$. This proves that H_t is relatively compact in C, and achieves the proof. □

4. Uniformly σ-additive sequences of probability measures

In this section we will prove tightness for a sequence of p.m, under the condition of uniform σ-additivity defined as follows:

4.1. A sequence (P_n) of p.m on a measurable space $(S, \mathcal{F})$ is said to be uniformly σ-additive if for every sequence $(A_k) \subset \mathcal{F}$ decreasing to ϕ we have:

$$\operatorname*{Lim}_k P_n(A_k) = 0, \quad \text{uniformly in } n \geqslant 1.$$

For example, if μ is a positive finite measure on $(S, \mathcal{F})$ such that $\operatorname{Lim}_{\mu(A) \to 0} P_n(A) = 0$, uniformly in $n \geqslant 1$, in which case we say that P_n is uniformly μ-continuous, then the sequence (P_n) is uniformly σ-additive.

Theorem 4.2. *Let S be a polish space (i.e. a metric complete separable space) with its Borel σ-field $\mathcal{B}_S$. Then every uniformly σ-additive sequence (P_n) of p.m on $\mathcal{B}_S$ is tight.*

Proof. Let $\varepsilon > 0$. Since each single p.m P_n is tight, by Theorem 1.4 in [1], for each n there is a compact set K_n of S such that $P_n(K_n) > 1 - \frac{\varepsilon}{2}$. Let $H_n = \bigcup_{j=1}^n K_j$ and $H = \bigcup_n K_n$. Then H_n is compact for each n and $H = H_k \cup (H \backslash H_k)$ for all k. Since $H \backslash H_k \searrow \phi$, we get by the uniform σ-additivity $P_n(H \backslash H_{k_0}) < \frac{\varepsilon}{2}$, for some k_0 and all n. Now we have $P_n(H) = P_n(H_{k_0}) + P_n(H \backslash H_{k_0}) < P_n(H_{k_0}) + \frac{\varepsilon}{2}$ and then $1 - \frac{\varepsilon}{2} < P_n(K_n) \leqslant P_n(H) < P_n(H_{k_0}) + \frac{\varepsilon}{2}$ for all n. So we deduce that:

$$H_{k_0} \text{ is compact} \quad \text{and} \quad P_n(H_{k_0}) > \varepsilon, \quad \text{for all } n.$$

This proves the tightness of the sequence (P_n). □

As a consequence we have:

Theorem 4.3. *If E is a separable Banach space, every uniformly σ-additive sequence (P_n) of p.m on C_E is tight.*

Proof. Since [0 1] and E are separable, the function space C_E is a polish space, so the result follows from Theorem 4.2. □

Now we turn to the weak convergence of p.m on C_E. We will need the notion of a determining class:

Definition 4.4. A family $\mathfrak{I}$ of Borel sets of C_E is called a determining class of the p.m on C_E if any two p.m that coincide on $\mathfrak{I}$ are identical.

For example any algebra generating the Borel σ-field is a determining class.

Consider for each $n \geqslant 1$ the product space E^n with the product topology, then the following is well known:

Proposition 4.5. *If E is separable, then the product σ-field of E^n is exactly its Borel σ-field.*

From now on, we assume that E is a separable Banach space.

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4.6. For each $n \geqslant 1$ and each finite set $s_1, s_2, \dots, s_n$ of points in [0 1], let $\tau_{s_1,s_2,\dots,s_n}$ be the projection from C_E onto E^n defined by: $\tau_{s_1,s_2,\dots,s_n}(X) = (X_{s_1}, X_{s_2}, \dots, X_{s_n})$. It is clear that $\tau_{s_1,s_2,\dots,s_n}$ is continuous, hence Borel measurable. A cylinder set of the space C_E is a set of the form $\tau^{-1}_{s_1,s_2,\dots,s_n}(A)$, for some $n \geqslant 1$, $s_1, s_2, \dots, s_n$ in [0 1] and some Borel set A of E^n. It is easy to see that the family of cylinder sets is an algebra on C_E. Moreover we have:

Proposition 4.7. *If E is separable, the algebra of cylinder sets generates the Borel σ-field of C_E. Consequently it is a determining class for the p.m on C_E.*

Proof. Let $\varepsilon > 0$, $X \in C_E$, and $n \geqslant 1$. Consider the following sets in C_E:

$$B(X,\varepsilon) = \Big\{Y \in C_E\colon \operatorname{Sup}_t \|X_t - Y_t\| \leqslant \varepsilon\Big\},$$

$$E_{n,X,\varepsilon} = \big\{Y \in C_E\colon \|X_{\frac{i}{n}} - Y_{\frac{i}{n}}\| \leqslant \varepsilon,\ i = 1, 2, \dots, n\big\}.$$

Then $B(X,\varepsilon)$ is a closed ball, $E_{n,X,\varepsilon}$ a cylinder set and we have $B(X,\varepsilon) = \bigcap_n E_{n,X,\varepsilon}$. So the σ-field generated by the cylinder sets contains the closed balls of C_E. Since C_E is separable, each open set is a countable union of closed balls and then belongs to this σ-field. □

Theorem 4.8. *Let E be a separable Banach space and let (P_n) be a tight sequence of p.m on C_E. Assume that all the finite dimensional distributions $P_n\tau^{-1}_{s_1,s_2,\dots,s_k}$ are weakly convergent. Then the sequence (P_n) itself is weakly convergent.*

Proof. For each k and each finite set $s_1, s_2, \dots, s_k$ of points in [0 1], there is a p.m $\lambda_{s_1,s_2,\dots,s_k}$ on E^k such that $P_n\tau^{-1}_{s_1,s_2,\dots,s_k} \Rightarrow \lambda_{s_1,s_2,\dots,s_k}$. On the other hand, by Theorem 1.2(a), each subsequence $P_{n'}$ of P_n contains a further subsequence $P_{n''}$ converging weakly to some p.m Q. But then we will have $P_{n''}\tau^{-1}_{s_1,s_2,\dots,s_k} \Rightarrow Q\tau^{-1}_{s_1,s_2,\dots,s_k}$, by the continuity of $\tau_{s_1,s_2,\dots,s_k}$. So we deduce that $Q\tau^{-1}_{s_1,s_2,\dots,s_k} = \lambda_{s_1,s_2,\dots,s_k}$ and since, Proposition 4.7, the cylinder sets form a determining class, the p.m Q is unique. By Theorem 2.3 of [1], $P_n \Rightarrow Q$. □

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